In Eqs. (6.5) and (6.6) the summation is carried out with respect to k, and for each k a sum is taken on the roots p_{ks} of the characteristic equation (6.4).

The expressions (6.5) and (6.6) permit us to draw the following conclusions of a general nature. Inasmuch as all the roots p of the characteristic equations (6.4) are located in the left half plane (except for the zero roots, which are of no interest), the sums in the expressions for U and V_α involve decaying vibratory solutions. Some time after the application of the impulse these components decay and only the free vibrations of the oscillators can be observed (these are determined by the last term in (6.5)). Comparison of this term with the expression (3.2) shows that the distribution of amplitudes of vibration of the oscillators along the bar coincides, except for a factor, with the distribution of the amplitudes of vibration of the points of the load-carrying medium for a steady excitation with frequency α.

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VARIATIONAL FORM OF THE EQUATIONS OF THE THEORY OF THERMODIFFUSION PROCESSES IN A DEFORMABLE SOLID

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The variational equation corresponding to the fundamental equations of thermoelasticity has been examined by Biot [1, 2], Balabukh and Shapovalov [3], and others [4, 5]. Sedov [6] and his disciples [7, 8] used variational methods to construct new models of continua.

A variational equation equivalent to the system of governing equations of a model which allows description of the interconnection between the deformation, heat and matter diffusion processes, and the most widespread types of boundary conditions, is presented herein.

1. Formulation of the question. Variational equation of the model. A deformed solid representing a two-component solid solution will be considered as a thermodynamic system whose state is determined by the temperature and entropy, chemical potential and particle concentration of the dissolved components, as well as the stress and strain tensors. We take its equilibrium as the initial state of such a system, and we shall henceforth examine the increments of the governing parameters and the parameters of the thermodynamic state of the medium relative to their values in the equilibrium state.

By selecting the strain e_{ij}, the entropy s and the concentration of dissolved component c_k as governing parameters, we introduce into the considerations a functional of the form

$$F = U(e_{ij}, s, c_k) + \sigma_{ij}[{}^1\!/_2(\nabla_j u_i + \nabla_i u_j) - e_{ij}] - t(\operatorname{div}\mathbf{S} + s) - \varphi_k(\operatorname{div}\mathbf{C}_k + c_k) +$$
$$+ p^2\left(\frac{\rho}{2} u_i u_i\right) + \frac{1}{2} p\left(\frac{T}{\varkappa_t}\mathbf{S}^2 + \frac{1}{\varkappa_c}\mathbf{C}^2\right) - X_i u_i \left(\nabla_{ij} = \frac{\partial u_j}{\partial x_i},\quad p = \frac{\partial}{\partial \tau}\right) \qquad (1.1)$$

Here $U(e_{ij}, s, c_k)$ is the integral energy per unit volume; σ_{ij} the stress tensor components; u_i the displacement vector components; t, φ_k the increments in temperature and chemical potential; $\mathbf{S}$, $\mathbf{C}_k$ entropy and concentration fluxes; ρ the density; $\varkappa_t$, $\varkappa_c$ the heat and mass conduction coefficients; X_i the volume force components; τ the time.

If the quantities u_i, e_{ij}, σ_{ij}, s, $\mathbf{S}$, c_k, $\mathbf{C}_k$, t, φ_k are considered independent, and their variations to be arbitrary, then the variational equation

$$\delta\Big\{\iiint\limits_V F\,dV - \iint\limits_{\Gamma_p} P_i' u_i\,d\Gamma - \iint\limits_{\Gamma_u} P_i(u_i - u_i')\,\partial\Gamma + \iint\limits_{\Gamma_t} (\mathbf{n}\cdot\mathbf{S})\,t'\,d\Gamma + \iint\limits_{\Gamma_{\varphi_k}} (\mathbf{n}\cdot\mathbf{C}_k)\,\varphi_k'\,d\Gamma +$$
$$+ \iint\limits_{\Gamma_{0t}} \frac{T}{2\varkappa_t}\,p\,[(\mathbf{n}\cdot\mathbf{S})^2]\,d\Gamma + \iint\limits_{\Gamma_{0\varphi_k}} \frac{1}{2\mu_c}\,p\,[(\mathbf{n}\cdot\mathbf{C}_k)^2]\,d\Gamma + \iint\limits_{\Gamma_{0t}} (\mathbf{n}\cdot\mathbf{S})\,t''\,d\Gamma + \iint\limits_{\Gamma_{0\varphi_k}} (\mathbf{n}\cdot\mathbf{C}_k)\,\varphi_k''\,d\Gamma\Big\} = 0 \qquad (1.2)$$

is equivalent to the following system of relationships:

$$\nabla_j\sigma_{ij} + X_i - \rho\frac{\partial^2 u_i}{\partial\tau^2} = 0, \qquad \frac{\partial}{\partial s}U(e_{ij}, s, c_k) - t = 0 \qquad (1.3)$$

$$\frac{T}{\varkappa_t}\frac{\partial\mathbf{S}}{\partial\tau} + \operatorname{grad} t = 0, \qquad \frac{\partial}{\partial c_k}U(e_{ij}, s, c_k) - \varphi_k = 0 \qquad (1.4)$$

$$\frac{1}{\varkappa_c}\frac{\partial\mathbf{C}_k}{\partial\tau} + \operatorname{grad}\varphi_k = 0, \qquad \frac{\partial}{\partial e_{ij}}U(e_{ij}, s, c_k) - \sigma_{ij} = 0 \qquad (1.5)$$

$$\operatorname{div}\mathbf{S} + s = 0, \qquad \operatorname{div}\mathbf{C}_k + c_k = 0 \qquad (1.6)$$

$$^1\!/_2(\nabla_j u_i + \nabla_i u_j) - e_{ij} = 0 \qquad (1.7)$$

and the boundary conditions

$$P_i = P_i' \quad \text{on } \Gamma_p, \qquad u_i = u_i' \quad \text{on } \Gamma_u \qquad (1.8)$$
$$t = t' \quad \text{on } \Gamma_t, \qquad \varphi_k = \varphi_k' \quad \text{on } \Gamma_{\varphi_k}$$

$$\frac{T}{\mu_t}\mathbf{n}\frac{\partial\mathbf{S}}{\partial\tau} + (t'' - t) = 0 \quad \text{on } \Gamma_{0t}, \qquad \frac{1}{\mu_c}\mathbf{n}\frac{\partial\mathbf{C}_k}{\partial\tau} + (\varphi_k'' - \varphi_k) = 0 \quad \text{on } \Gamma_{0\varphi_k} \qquad (1.9)$$

Here μ_t and μ_c are coefficients characterizing the heat and mass exchange from the body surface, and the portions of the bounding surface are selected in such a manner that:

$$\Gamma_p + \Gamma_u = \Gamma_{0t} + \Gamma_t = \Gamma_{0\varphi_k} + \Gamma_{\varphi_k} = \Gamma$$

where Γ is the total surface of the body occupying the volume V.

2. Analysis of the relationships obtained. The relationships presented in Sect. 1 contain derivatives of the internal energy with respect to the governing parameters. Therefore, these equations can be written explicitly only for a concrete assignment of the internal energy as an explicit function of their arguments.

Upon selecting e_{ij}, s, c_k as governing parameters, the internal energy of the system is a scalar characteristic function of these parameters. Hence, it should be independent of processes of the system making the transition from one state into another and of the choice of coordinate system. It should thus be expressed only in terms of invariants formed from the governing parameters.

Starting from these considerations, let us represent the internal energy as follows:

$$U(e_{ij}, s, c_k) = \frac{T}{2\omega^{e, c_k}} s^2 + \frac{1}{2} d_{c_k}^{e, s} c_k^2 + G_{s, c_k} e_{ij}^2 + \frac{1}{2} \lambda_{s, c_k} e_{ii}^2 + \\ + d_s^{e, c_k} s c_k + (\alpha_s^{e, c_k} s + \alpha_{c_k}^{e, s} c_k) e_{ii} + U_0 \tag{2.1}$$

where U_0 is the value of the internal energy in the initial state, T is the absolute temperature of the system, and the physical meaning of the other constants in (2.1) is to be explained later.

The second equations in (1.3)–(1.5) hence become

$$\left(\frac{\partial U}{\partial s}\right)_{e_{ij}, c_k} = t = \frac{T}{\omega^{e, c_k}} s + d_s^{e, c_k} c_k + \alpha_s^{e, c_k} e \tag{2.2}$$

$$\left(\frac{\partial U}{\partial c_k}\right)_{e_{ij}, s} = \varphi_k = d_{c_k}^{e, s} c_k + d_s^{e, c_k} s + \alpha_{c_k}^{e, s} e \tag{2.3}$$

$$\left(\frac{\partial U}{\partial e_{ij}}\right)_{s, c_k} = \sigma_{ij} = 2G_{s, c_k} e_{ij} + \lambda_{s, c_k} e \delta_{ij} + (\alpha_s^{e, c_k} s + \alpha_{c_k}^{e, s} c_k) \delta_{ij} \tag{2.4}$$

Now, let us utilize relationships (2.2)–(2.4) which represent the equation of state to clarify the physical meaning of the material characteristics in (2.1).

The constant ω^{e, c_k} is the specific heat per unit volume of material for fixed $e = e_{ii}$, c_k. Indeed, from (2.2) we have

$$\omega^{e, c_k} = T \left(\frac{\partial s}{\partial t}\right)_{e, c_k} = \left(\frac{\delta Q}{dt}\right)_{e, c_k} \tag{2.5}$$

where δQ is the quantity of heat acquired by unit volume when the temperature changes by dt.

From (2.3) we find that

$$d_{c_k}^{e, s} = \left(\frac{\partial \varphi_k}{\partial c_k}\right)_{e, s}, \qquad d_s^{e, c_k} = \left(\frac{\partial \varphi_k}{\partial s}\right)_{e, c_k}$$

Therefore, these constants characterize the change in chemical potential with the change, respectively, in concentration c_k and entropy s for a given material.

Analogously, we obtain from (2.4)

$$\delta_{ij} \alpha_{c_k}^{e, s} = \left(\frac{\partial \sigma_{ij}}{\partial c_k}\right)_{s, e}, \qquad \delta_{ij} \alpha_s^{e, c_k} = \left(\frac{\partial \sigma_{ij}}{\partial s}\right)_{e, c_k}$$

The $\alpha_{c_k}^{e,s}$, α_s^{e,c_k} are defined as material constants characterizing the change in stress with the change in concentration c_k and entropy s by these relationships. The relationships

$$\left(\frac{\partial \sigma_{ij}}{\partial e_{ij}}\right)_{s, c_k} = 2G_{s, c_k} + \lambda_{s, c_k} \delta_{ij}$$

obtained from (2.4), define the constants G_{s,c_k}, λ_{s,c_k} as the elastic characteristics of the material for constant physical parameters s, c_k.

If the conjugates σ_{ij}, t, φ_k are selected to be governing instead of the parameters

e_{ij}, s, c_k (all or some of these), then other material characteristics associated with the relationships defined above presented in [9] will occur in equations of the type (2.2) – –(2.4). Analysis of the first equations in (1.4), (1.5) is carried out for the case of a two-component solid solution in order to simplify the mathematical transformations.

Solving (2.2), (2.3) for s and c, we find

$$s = \frac{\omega^{e,\varphi}}{T} t - \alpha_t^{e,\varphi} e - \frac{d_t^{e,c}}{d_c^{e,t}} \varphi, \quad c = \frac{1}{d_c^{e,t}} \varphi - \alpha_\varphi^{e,t} e - \frac{d_t^{e,c}}{d_c^{e,t}} t \tag{2.6}$$

After having been operated upon by a Hamilton operator, and (1.6) and (2.6) have been taken into account, (1.4), (1.5) take the following form:

$$\varkappa_t \Delta t = T \frac{\partial}{\partial \tau} \left[\frac{\omega^{e,\varphi}}{T} t - \alpha_t^{e,\varphi} e - \frac{d_t^{e,c}}{d_c^{e,t}} \varphi \right], \qquad \varkappa_c \Delta \varphi = \frac{\partial}{\partial \tau} \left[\frac{1}{d_c^{e,t}} \varphi - \alpha_\varphi^{e,t} e - \frac{d_t^{e,c}}{d_c^{e,t}} t \right] \tag{2.7}$$

which have the meaning of generalized equations of heat conduction and diffusion of matter.

It should be noted that the first relationships of (1.4), (1.5) can be interpreted as the kinetic equations. Indeed, if it is taken into account that

$$T \frac{\partial S}{\partial \tau} = \mathbf{J}_q, \qquad \frac{\partial C}{\partial \tau} = \mathbf{J}_m$$

then these relationships can be reduced to

$$\mathbf{J}_q = -\varkappa_t \operatorname{grad} t, \qquad \mathbf{J}_m = -\varkappa_c \operatorname{grad} \varphi \tag{2.8}$$

where $\mathbf{J}_q$, $\mathbf{J}_m$ are the heat and matter fluxes.

Therefore, the variational equation (1.2) is equivalent to the equation of motion (1.3), the generalized equations of heat conduction and the diffusion of matter (the first equations in (1.4), (1.5)), the equations of state (the second equations in (1.3)–(1.5)), the balance equations (1.6), the geometric relationships (1.7), as well as boundary conditions of the type (1.8) or (1.9).

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